

# **TOWARDS A THEORY OF QUANTUM BLACK HOLE** VICTOR BEREZIN Institute for Nuclear

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## **Abstract**

We describe some specific quantum black hole model. It is pointed out that the origin of a black hole entropy is the very process of quantum gravitational collapse. The quantum black hole mass spectrum is extracted from the mass spectrum of the gravitating source. The classical analog of quantum black hole is constructed.

## Introduction

What does urge a researcher to investigate quantum black holes? Honestly, his own intrinsic interest. Besides, there are other, more (or less) scientific reasons. It is commonly believed that only small black holes can be considered as quantum objects. Small, what does it mean? To estimate, we should compare the size of the black hole with the corresponding Compton length. The gravitational radius  $r_g$  of the black hole of mass  $m$  equals  $r_g = \frac{2Gm}{c^2}$ , where  $G$  is the Newtonian gravitational constant, and  $c$  is the speed of light. The Compton length of such particle is  $\lambda = \frac{\hbar}{mc}$  ( $\hbar$  is the Planck's constant). If  $r_g \simeq \lambda$ , then the so called Planckian mass is  $m_{pl} = \sqrt{\frac{\hbar c}{G}} \sim 10^{-5} gr$ , the Planckian length is  $l_{pl} = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-33} cm$ . In what follows we use the units in which  $\hbar = c = 1$ , so  $m_{pl} = 1/\sqrt{G}$ ,  $l_{pl} = \sqrt{G}$ . Black holes of so small mass and sizes could be created from large metric fluctuation in the very early Universe (primordial black holes), or during vacuum phase transition.

In classical General Relativity black holes are very special (and, therefore, very interesting) objects. First of all, they are universal in the sense that they are described by only few parameters: their mass, angular momenta and charges. In the process of a black hole formation, (i.e., in the process of gravitational collapse) all higher momenta and non conserved charges are radiating away. This feature is formulated as a following conjecture: "black holes has no hairs". Thus, a black hole formation results generically in the loss of information about initial states and previous history of collapsing matter. The boundary of the black hole, the so-called event horizon, is the null hypersurface that acts as a one-way membrane. The matter can fall inside but can not go outside. Because of this the area of black hole horizon can not decrease. These two features, the loss of information and non decrease of the horizon area, allowed J.Bekenstein [2] to suggest the analogy between the black hole physics and thermodynamics and identify the area of the horizon with the entropy (up to some factor). He did this for the simplest, spherically symmetric neutral (Schwarzschild) black hole which characterized by only one parameter, Schwarzschild mass. Later the four laws of thermodynamics were derived for a general black hole.

In thermodynamics the appearance of the entropy is accompanied by the temperature. While the nature of the black hole entropy was more or less

clear, the notion of its temperature remained mysterious until the revolutionary work by S.Hawking [4]. He showed that the black hole temperature introduced by J.Bekenstein is the real temperature in the sense that the black hole radiates, and this radiation has a Planck's spectrum. The entropy appeared equal one fourth of the event horizon area divided by Planckian length squared. Thus, even large (compared to the Planckian mass and size) black holes exhibit quantum features. It should be stressed that such quantum effect is global, namely, it emerges as a result of nontrivial boundary conditions for the wave function of a quantum field theory in curved space-times nontrivial causal structure (existence of the event horizon(s)).

Due to the process of Hawking's evaporation any (even super-large) black hole becomes eventually small enough to be considered as a (local) quantum object. At this stage the combination of the global and local quantum effects may result in unpredictable features. That why it is so exciting to try to understand quantum black hole physics.

## Classical Model

Everybody knows what the classical black hole is. In short, black hole is a region of a space-time manifold beyond an event horizon. In turn, an event horizon is a null surface that separates the region from which null geodesics can escape to infinity and that one from which they cannot. It is important to stress that the notion of the event horizon is global, it requires knowledge of both past and future histories. In classical physics we have trajectories of particles, we have geodesics, so, everything can be, in principle, calculated. In quantum physics there are no trajectories and the event horizon can not be defined. Thus, we have to seek for quite a different definition of a quantum black hole. Till now we have no consistent theory of quantum gravity. All this forces us to start with considering some models. The simpler, the better.

The simplest is the so-called Schwarzschild eternal black hole. Its geometry is a geometry of non-traversable wormhole. There are two asymptotically flat regions at spatial infinities connected by the Einstein-Rosen bridge. The gravitating source is concentrated at two spacelike singular surfaces or zero radius. Two sides of the Einstein-Rosen bridge are causally disconnected

and separated by event horizons. The narrowest part of the bridge is called a throat, its size is the size of the horizon. Eternal black holes are parameterized by total (Schwarzschild) mass of the system. This one-parameter family is the only spherically symmetric solution to the vacuum Einstein equations. The spherically symmetric gravity can be fully quantized in the mini-superspace (frozen) formalism [5, 6]. The result of such quantization is trivial, quantum functional depends only on Schwarzschild mass. Physically it is quite understandable. Indeed, one allows the matter sources first to collapse classically and then starts to quantize such a system. What is left for quantization? Nothing. Mathematically, eternal black holes has no dynamical degrees of freedom. No real gravitons (because of frozen spherical symmetry), no matter source motion.

To get physically meaningful result we need to introduce some dynamical gravitating source. The simplest generalization of the point mass is the spherically symmetric self-gravitating thin dust shell. The theory of thin shells was developed by W.Israel [7] and applied to various problems by many authors. For simplicity we consider the case when the shell is the only source of gravitational field. Then, inside the shell the space-time is flat, and outside it is some part of Schwarzschild solution. The dynamics of such dust shell is completely described by the single equation

$$\sqrt{\dot{\rho}^2 + 1} - \sigma \sqrt{\dot{\rho}^2 + 1 - \frac{2Gm}{\rho}} = \frac{GM}{\rho} \quad (1)$$

where  $\rho$  is the radius of the shell as a function of proper time of an observer sitting on the shell, a dot denotes the proper time derivative,  $m$  is the total (Schwarzschild) mass of the shell, and  $M$  is the bare mass (e.g., the sum of the masses of constituent dust particles without gravitational mass defect). The quantify  $\sigma$  is the sign function distinguishing two different types of shells. If  $\sigma = +1$ , the shells moves on “our” side of the Einstein-Rosen bridge and the radii increase when one goes in the outward direction of the shell. We will call this the black hole case. If  $\sigma = -1$ , the shell moves beyond the event horizon on the other side or the Einstein-Rosen bridge, and radii out of the shell first start to decrease, reach the minimal value at the throat and start to increase already on our side of the bridge. We will call this the wormhole-like case (such a configuration is also called a semi-closed world). In what follows we confine ourselves by considering the bound motion only. It can be shown

that

$$\begin{aligned} \frac{m}{M} &> \frac{1}{2} & \text{if} & \quad \sigma = +1 \\ \frac{m}{M} &< \frac{1}{2} & \text{if} & \quad \sigma = -1 \end{aligned} \quad (2)$$

The two types of shells can be distinguished by different signs of the following inequality ( $\rho_0$  is the radius of the shell at the turning point)

$$\begin{aligned} \frac{\partial m}{\partial M} &> 0 & \text{if} & \quad \sigma = +1 \\ \frac{\partial m}{\partial M} &< 0 & \text{if} & \quad \sigma = -1 \end{aligned} \quad (3)$$

The seemingly unusual sign in the wormhole case can be easily explained. Indeed, the large the bare mass  $M$  of the shell, the stronger its gravitational field, the more narrow, therefore, the throat, and, consequently, the smaller the total mass  $m$  of the system.

## Quantum Model

The spherically symmetric space-times with shells can also be fully quantized in the mini-superspace formalism [9]. All the quantum constraints can be solved, except one. This is the Hamiltonian constraint or, Wheeler-DeWitt equation, for the shell (here we write it only for the case of bound motion)

$$\Psi(s + i\zeta) + \Psi(s - i\zeta) = \frac{2 - \frac{1}{\sqrt{s}} - \frac{M^2}{4m^2s}}{(1 - \frac{1}{\sqrt{s}})^{1/2}} \Psi(s) \quad (4)$$

Here  $s$  is a dimensionless radius squared (normalized by the horizon area,  $s = R^2/R_g^2 = R^2/4G^2m^2$ ),  $\zeta = \frac{1}{2}(\frac{mpl}{m})^2$ , and  $i$  is the imaginary unit. The Eqn.(4) is an equation in finite differences, and the shift in the argument is pure imaginary. Thus, the “good” solutions should be analytical functions. Besides, there are branching points at the horizons (in our case at  $s = 1$ ). Thus, the wave functions should be analytical on a Riemann’s surface with a two leaves. The physical reason to consider two Riemann’s surface is the

following. In quantum theory there are no trajectories. Thus, even if a shell has parameters  $m$  and  $M$  (total and bare mass) corresponding to the black hole (or wormhole) case, its wave function is, in general, everywhere nonzero, “feel” both infinities on both sides of Einstein-Rosen bridge. The analyticity requirement is so stringent that there is no need to solve the quantum equation in order to calculate a mass spectrum. One should investigate only a behavior of solutions in the vicinity singular points (infinities and singularities) and around branching points, and then to compare these asymptotics. In such a way the following quantum conditions were found for a discrete mass spectrum in the case of bound motion [9].

$$\frac{2m^2 - M^2}{\sqrt{M^2 - m^2}} = \frac{2m_{pl}^2}{m}n \quad (5)$$

$$M^2 - m^2 = 2m_{pl}^2(1 + 2p)$$

where  $n$  and  $p$  are integers. The appearance of two quantum conditions instead of only one in conventional quantum mechanics is due to a nontrivial causal structure of Schwarzschild manifold (two infinities!).

Let us discuss some properties of the spectrum that arises from these conditions.

1. For larger values of quantum number  $n$  ( $\frac{M^2}{m^2} - 1 \ll 0$ ) one can easily derive nonrelativistic Rydberg formula for Kepler’s problem,  $E_{nonrel} = M - m = -\frac{G^2 M^4}{8n^2}$ .

2. The role of turning point  $\rho_0$  is now played by the integer  $n$ . Thus, keeping  $n$  constant and calculating  $\gamma = \frac{\partial m}{\partial M}|_n$  one can distinguish between a black hole case ( $\gamma > 0$ ) and a wormhole case ( $\gamma < 0$ ). It appears that  $\frac{\partial m}{\partial M}|_n > 0$  for  $n \geq n_0$ , negative or zero, and

$$|n_0| = E[\sqrt{2}\sqrt{13\sqrt{5} - 29}(1 + 2p)] \quad (6)$$

3. There exists a minimal possible value for a black hole mass. This occurs if  $p = n_0 = 0$ ,

$$m_{min} = \sqrt{2}m_{pl} \quad (7)$$

4. The spectrum described by Eqn.(6) is not universal in the sense that corresponding wave functions form a two-parameter family  $\Psi_{n,p}(R)$ .

But for quantum Schwarzschild black hole we expect a one-parameter family of wave functions. Quantum black holes should have no hairs, otherwise there will be no smooth limit to the classical black holes. All this means

that our spectrum is not a quantum black hole spectrum, and our shell does not collapse (like an electron in hydrogen atom). Physically, it is quite understandable, because the radiation is yet included into consideration.

And again, we will use thin shells to model the radiation, but this time shells should be null. Let  $m_{in}$  and  $m_{out}$  be a Schwarzschild mass inside and outside the shell. Then, the quantum constraint equation reads as follows [10]

$$\Psi(m_{in}, m_{out}, s - i\zeta) = \sqrt{\frac{1 - \frac{\mu}{\sqrt{s}}}{1 - \frac{1}{\sqrt{s}}}} \Psi(m_{in}, m_{out}, s) \quad (8)$$

here  $\mu = m_{in}/m_{out}$ ,  $\zeta = \frac{1}{2}m_{pl}^2/m_{out}^2$ . The existence of the second infinity on the other side of the Einstein-Rosen bridge leads to the following quantization condition ( $m = m_{out}$ )

$$\delta m = m_{out} - m_{in} = -2m + 2\sqrt{m^2 + km_{pl}^2}, \quad (9)$$

where  $k$  is an integer. It is interesting to note that if we put  $k = 1$  (minimal radiating energy) and require  $\delta m < m$  (not more than the total mass can be radiated away), then we obtain

$$m = m_{out} > \frac{2}{\sqrt{5}}m_{pl}. \quad (10)$$

Thus, the black hole with the mass given by Eqn.(7) is not radiating and, therefore, it can not be transformed into semi-closed world (wormhole-like case).

The discrete spectrum of radiation (9) is universal in the sense that it does not depend on the structure and mass spectrum of the gravitating source. This means that the energies of radiating quanta do not coincide with level spacing of the source. The most natural way in resolving such a paradox is to suppose that quanta are created in pairs. One of them is radiated away, while another one goes inside. Thus, the quantum collapse can not proceed without radiating even in the case of spherical symmetry. This radiation is accompanying with creation of new shells inside the primary shell we started with. We see, that the internal structure of quantum black hole is formed during the very process of quantum collapse. And if at the beginning we had one shell and knew everything about it, then already after the first pulse of radiation we have more than one way of creating the inner quantum.

So, initially the entropy of the system was zero, it starts to grow during the quantum collapse. If somehow such a process would stop we would call the resulting object “a quantum black hole”. The natural limit is the transition from black hole to the wormhole-like shell. The matter is that such a transition requires (at least in quasi-classical regime) insertion of an infinitely large volume, and the quasiclassical probability for this process is zero.

Let us write down the spectrum of the shell with nonzero Schwarzschild mass, the total mass inside,  $m_{in} \neq 0$

$$\begin{aligned} \frac{2(\Delta m)^2 - M^2}{\sqrt{M^2 - (\Delta m)^2}} &= \frac{2m_{pl}^2}{\Delta m + m_{in}} n \\ M^2 - (\Delta m)^2 &= 2(1 + 2p)m_{pl}^2 \end{aligned} \quad (11)$$

Here  $\Delta m$  is the total mass of the shell,  $M$  is the bare mass, the total mass of the system equals  $m = m_{out} = \Delta m + m_{in}$ . For the black hole case  $M^2 < 4m\Delta m$ , or

$$\frac{\Delta m}{M} > \frac{1}{2} \left( \sqrt{\left(\frac{m_{in}}{M}\right)^2 + 1} - \frac{m_{in}}{M} \right). \quad (12)$$

After switching on the process of radiation governed by Eqn.(9), the quantum collapse starts. Our computer simulations shows that evolves in the “correct” direction, e.g. it becomes nearer and nearer to the threshold (12) between the black hole case and wormhole case. The process stops exactly at  $n = 0$ !

The point  $n = 0$  in the spectrum is very special. Only in such a state the shell does not “feel” not only the outer regions (what is natural for the spherically symmetric configuration) but it does not know anything about what is going on inside. It “feel” only itself. Such a situation reminds the classical (non-spherical) collapse. Finally when all the shells (both the primary one and newly produced) are in the corresponding states  $n_i = 0$ , the system does not “remember” its own history. And this is a quantum black hole. The masses of all the shells obey the relation

$$\Delta m_i = \frac{1}{\sqrt{2}} M_i. \quad (13)$$

The subsequent quantum Hawking’s evaporation can produced only via some collective excitations and formation, e.g., of a long chain of microscopic semi-closed worlds.



## Classical analog of quantum black hole

Let us consider large ( $m \gg m_{pl}$ ) quantum black holes. The number of shells (both primary ones and created during collapse) is also very large, and one may hope to construct some classical continuous matter distribution that would mimic the properties of quantum black holes. First of all, we should translate the “no memory” state ( $n = 0$  for all the shells) into “classical language”. To do this let us rewrite the Eqn.(1) (energy constraint equation) for the shell, inside which there is some gravitating mass  $m_{in}$ ,

$$\sqrt{\dot{\rho}^2 + 1 - \frac{2Gm_{in}}{\rho}} - \sqrt{\dot{\rho}^2 + 1 - \frac{2Gm_{out}}{\rho}} = \frac{GM}{\rho} \quad (14)$$

and consider a turning point, ( $\dot{\rho} = 0$ ,  $\rho = \rho_0$ ):

$$\Delta m = m_{out} - m_{in} = M \sqrt{1 - \frac{2Gm_{in}}{\rho_0}} - \frac{GM^2}{2\rho_0}. \quad (15)$$

It is clear now that in order to make parameters of the shell ( $\Delta m$  and  $M$ ) not depending on what is going on inside we have to put  $m_{in} = a\rho_0$ .

Our quantum black hole is in a stationary state. Therefore, a classical matter distribution should be static. We will consider a static perfect fluid with energy density  $\varepsilon$  and pressure  $p$ . A static spherically symmetric metric can be written as

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad (16)$$

where  $\nu$  and  $\lambda$  are functions of the radial coordinate  $r$  only. The relevant Einstein's equations are (prime denotes differentiation in  $r$ )

$$\begin{aligned} 8\pi G\varepsilon &= -e^\lambda \left( \frac{1}{r^2} - \frac{\lambda'}{r} \right) + \frac{1}{r^2}, \\ -8\pi Gp &= -e^\lambda \left( \frac{1}{r^2} - \frac{\nu'}{r} \right) + \frac{1}{r^2}, \\ -8\pi Gp &= -\frac{1}{2}e^\lambda \left( \nu'' + \frac{\nu'^2}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu'\lambda'}{2} \right) \end{aligned} \quad (17)$$

The first of these equations can be integrated to yield

$$e^{-\lambda} = 1 - \frac{2Gm(r)}{r}, \quad (18)$$

where

$$m(r) = 4\pi \int_0^r \varepsilon r'^2 dr' \quad (19)$$

is the mass function, that must be identified with  $m_{in}$ . Thus,  $m(r) = ar$ , and

$$\varepsilon = \frac{a}{4\pi r^2}, \quad e^{-\lambda} = 1 - 2Ga. \quad (20)$$

We can also introduce a bare mass function

$$M(r) = 4\pi \int_0^r \varepsilon e^{\frac{\lambda}{2}} r'^2 dr', \quad (21)$$

and from Eqn.(20) we get

$$M(r) = \frac{ar}{\sqrt{1 - 2Ga}} \quad (22)$$

The remaining two equations can now be solved for  $p(r)$  and  $e^\nu$ . The solution for  $p(r)$  that has the correct nonrelativistic limit is

$$p(r) = \frac{b}{4\pi r^2}, \quad b = \frac{1}{G}(1 - 3Ga - \sqrt{1 - 2Ga}\sqrt{1 - 4Ga}), \quad (23)$$

and for  $e^\nu$  we have

$$e^\nu = Cr^{2G\frac{a+b}{1-2Ga}}. \quad (24)$$

The constant of integration  $C$  can be found from matching of the interior and exterior metrics at some boundary  $r = r_0$ . Let us suppose that  $r > r_0$  the space-time is empty, so the interior should be matched to the Schwarzschild metric. Of course, to compensate the jump in pressure ( $\Delta p = p(r_0) = p_0$ ) we must introduce some surface tension  $\Sigma$ . From matching conditions (see, e.g. [8]) it follow that

$$\begin{aligned} C &= (1 - 2Ga)r_0^{-2G\frac{a+b}{1-2Ga}}, \\ e^\nu &= (1 - 2Ga)\left(\frac{r}{r_0}\right)^{2G\frac{a+b}{1-2Ga}}, \\ \Sigma &= \frac{2\Delta p}{r_0} = \frac{b}{2\pi r_0^3} \end{aligned} \quad (25)$$

We would like to stress that the pressure  $p$  in our classical model is not real but only effective because it was introduce in order to mimic the quantum stationary states. We see, that the coefficient  $b$  in Eqn.(23) becomes a

complex number if  $a > 1/4G$ . Hence, we must require  $a \leq 1/4G$ , and in the limiting point we have the stiffest possible equation of state  $\varepsilon = p$ . It means also that hypothetical quantum collective excitations (phonons) would propagate with the speed of light and could be considered as massless quasiparticles. It is remarkable that in the limiting point we have  $m(r) = M(r)/\sqrt{2}$  - the same relation as for the total and bare masses in the “no memory” state  $n = 0$ ! The total mass  $m_0 = m(r_0)$  and the radius  $r_0$  in this case are related  $m_0 = 4Gr_0$  - twice the horizon size.

Calculations of Riemann curvature tensor  $R_{klm}^i$  and Ricci tensor  $R_{ik}$  show that if  $p < \varepsilon$  ( $a \neq b$ ) there is a real singularity at  $r = 0$ . But, surprisingly enough, both Riemann and Ricci tensors have finite limits at  $r \rightarrow 0$ , if  $\varepsilon = p$  ( $a=b=1/4G$ ). Therefore we are allowed to introduce the so-called topological temperature in the same way as for classical black holes. The recipe is the following. One should transform the space-time metric by the Wick rotation to the Euclidean form and smooth out the canonical singularity by the appropriate choice of the period for the imaginary time coordinate. The imaginary time coordinate is considered proportional to some angle coordinate. In our case the point  $r = 0$  is already the coordinate singularity. The azimuthal angle  $\phi$  has the period equal to  $\pi$ . Thus, all other angles should be periodical with the period  $\pi$ . The topological temperature is just the inverse of this period.

The easy exercise shows, that the temperature

$$T = \frac{1}{2\pi r_0} = \frac{1}{8\pi G m_0} = T_{BH} \quad (26)$$

exactly the same as the Hawking's temperature  $T_{BH}$  [4]! The very possibility of introducing a temperature provides us with the one-parameter family of models with universal distributions of energy density and pressure

$$\varepsilon = p = \frac{1}{16\pi G r^2}, \quad (27)$$

the parameter being the total mass  $m_0$  or the size  $r_0 = 4Gm_0$ .

We can now develop some thermodynamics for our model. First of all we should distinguish between global and local thermodynamic quantities. The global quantities are those measured by a distant observer. He measures the total mass of the system  $m_0$  and the black temperature  $T_{BH} = T_\infty$  and does not know anything more. Let us assume that this observer is rather educated

in order to recognize he is dealing with a black hole and to write the main thermodynamic relation

$$dm = TdS. \quad (28)$$

In this way he ascribes to a black hole some amount of entropy, namely, the Hawking-Bekenstein value [2, 4]

$$S = \frac{1}{4} \frac{(4\pi r_g)^2}{l_{pl}^2} = 4\pi G m_0^2 = 4\pi \left(\frac{m_0}{M_{pl}}\right)^2 \quad (29)$$

The local observer who measure distribution of energy, pressure and local temperature is also rather educated and writes quite a different thermodynamic relation

$$\varepsilon(r) = T(r)s(r) - p(r) - \mu(r)n(r). \quad (30)$$

Here  $\varepsilon(r)$  and  $p(r)$  are energy density and pressure,  $T(r)$  is the local temperature distribution,  $s(r)$  is the entropy density,  $\mu(r)$  is the chemical potential, and  $n(r)$  is the number density of some (quasi)"particles". For the energy density and pressure the local observer gets, of course, the relation (27), and for the temperature - the following distribution

$$T(r) = \frac{1}{\sqrt{2}\pi r}, \quad (31)$$

which is compatible with the law  $T(r)e^{\frac{v}{2}} = \text{const}$  and the boundary condition  $T_\infty = T_{BH}$ . Such a distribution is remarkable in that if some outer layer of our perfect fluid would be removed, the inner layers would remain in thermodynamic equilibrium. And what about the entropy density? Surely, the local observer is unable to measure it directly but he can receive some information concerning the total entropy from the distant observer. This information and the measured temperature distribution (31) allows him to deduce that

$$S(r) = \frac{1}{8\sqrt{2}Gr} \quad (32)$$

and

$$S(r)T(r) = \frac{1}{16\pi Gr^2} \quad (33)$$

It is interesting to note that in the main thermodynamic equation the contribution from the pressure is compensated exactly by the contribution

from the temperature and entropy. It is noteworthy to remind that the pressure in our classical analog model is of quantum mechanical origin as well as the black hole temperature. And what is left actually is the dust matter we started from in our quantum model, namely,

$$\varepsilon = \mu n = \frac{1}{16\pi G r^2} \quad (34)$$

We may suggest now that the quantum black hole is the ensemble of some collective excitations, the black hole phonons, and  $n(r)$  is just the number density of such phonons. The chemical potential obeys the relation  $\mu e^{\frac{\mu}{T}} = \text{const}$  [?]. Hence,  $\mu \sim 1/r$ , and for the number density we can write

$$n = \frac{1}{32\pi G \alpha^2 r} \quad (35)$$

where  $\alpha$  is some numerical coefficient. By integrating over the volume we get for number of phonons

$$N = 4\pi \int_0^{r_0} n r^2 dr = \frac{r_0^2}{16\pi G \alpha^2} = \frac{G m_0^2}{\alpha^2} = \frac{m_0^2}{\alpha^2 m_{pl}^2} \quad (36)$$

Thus, we obtained the famous Bekenstein-Mukhanov spectrum

$$m_{BH} = \alpha m_{pl} \sqrt{N}. \quad (37)$$

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